A Trust Region Strategy For Equality Constrained Optimization[1]

by

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Abstract

Many current algorithms for nonlinear constrained optimization problems determine a direction by solving a quadratic programming subproblem. The global convergence properties are addressed by using a line search technique and a merit function to modify the length of the step obtained from the quadratic program.

In unconstrained optimization, trust regions strategies have been very successful. In this paper we present a new approach for equality constrained optimization problems based on a trust region strategy. The direction selected is not necessarily the solution of the standard quadratic programming subproblem.

Key words

Nonlinear programming, global convergence, trust region methods.

1. Introduction.

Consider the equality constrained optimization problem :

$$\text{(NLE)} \qquad \begin{array}{l} \textit{minimize } f(x) \\ \textit{subject to } g(x) = 0, \end{array}$$

where $f : \mathrm{R}^n \rightarrow \mathrm{R}$ and $g : \mathrm{R}^n \rightarrow \mathrm{R}^m$ $(m \leq n)$. It is assumed that the problem functions are at least twice continuously differentiable, that a solution exists, and that $\nabla g(x)$ has full rank.

Several authors including Fletcher [2], Gay [3], and Sorensen [12], have considered a trust region approach for optimization problems with linear constraints. From a theoretical point of view, the extension from unconstrained optimization to linearly constrained optimization is somewhat straightforward; one merely focuses attention on the subspace of interest. For nonlinear constraints the extension is not at all clear. The main attempt in this area has been Vardi [14]. While this work contains some interesting results, it leaves several important questions unanswered. Our objective is to develop an effective trust region algorithm for problem NLE.

2. Motivation for Our Approach.

One of the more successful methods for solving problem NLE is the successive quadratic programming (SQP) approach where, at each iteration, the step is calculated as the solution of the quadratic programming problem :

$$\text{(QP)} \qquad \begin{array}{l} \textit{minimize } \ q_{QP}(s) = \nabla_x L(x,\lambda)^T s + \frac{1}{2} s^T B s \\ \textit{subject to } \ g(x) + \nabla g(x)^T s = 0 \ , \end{array}$$

where $\nabla_x L(x,\lambda)$ is the gradient of the Lagrangian function

$$L(x,\lambda) = f(x) + \lambda^T g(x),$$

$\lambda \in \mathrm{R}^m$, and B is an approximation to $\nabla^2_{xx} L(x,\lambda)$. The step for the multiplier λ is obtained as the multiplier associated with the solution of problem QP.

The most natural way to introduce the trust region idea is to add a constraint which restricts the size of the step in problem QP, see Vardi [14]. However, this approach may lead to inconsistent constraints, and it is not clear how to overcome this problem. Instead of adding the trust region constraint to the standard QP problem, we consider adding it to a somewhat different problem.

Suppose we want to solve $g(x)=0$ using a standard trust region method. We have a current point x_c and a bound Δ_c on the length of the step we are willing to take from x_c. At each iteration the step is calculated by solving :

$$\begin{aligned} &\mathit{minimize} \ \tfrac{1}{2} \| \, g(x_c) + \nabla g(x_c)^T s \, \|_2^2 \\ &\mathit{subject\ to} \ \| \, s \, \|_2 \leqq \Delta_c \end{aligned}$$

where $\| \ \|_2$ denotes the 2-norm.

If the algorithm simply took the "best steepest descent step", i.e. the Cauchy step s_{CP}, then under reasonable assumptions global convergence can be demonstrated, see Powell [10], Moré and Sorensen [7], and Schultz, Schnabel and Byrd [11]. That is, as long as the step s satisfies

$$\|g(x_c)+\nabla g(x_c)^T s\|_2^2 \ \leqq \ \|g(x_c)+\nabla g(x_c)^T s_{CP}\|_2^2,$$

convergence to a solution of $g(x) = 0$ is obtained. This fact is the basis for our approach.

Define the set Y as

$$\begin{aligned} Y = \{ \, s : \ &\|s\|_2 \leqq \Delta_c \ \text{and} \\ &\|g(x_c)+\nabla g(x_c)^T s\|_2^2 \leqq \|g(x_c)+\nabla g(x_c)^T s_{CP}\|_2^2 \, \} \, . \end{aligned}$$

That is, Y is the set of steps from x_c that are inside the trust region and give at least as much descent on the 2-norm of the residuals of the linearized constraints as the Cauchy step, (see Figure 1). By choosing any point in Y we will generate a sequence which is guaranteed to converge to a feasible point. We take advantage of this freedom by choosing an s which minimizes a quadratic model, $q(s)$, of the objective function f over Y. The step is calculated by solving the problem :

$$\begin{aligned} &\mathit{minimize} \ \ q_c(s) \\ &\mathit{subject\ to} \ \ \| \, s \, \|_2 \leqq \Delta_c \\ &\qquad\qquad\qquad \| \, g(x_c) + \nabla g(x_c)^T s \, \|_2^2 \leqq \theta_c \, , \end{aligned}$$

where $q_c(s)$ is a quadratic approximation to the function f and $\theta_c \ = \ \| \, g(x_c) + \nabla g(x_c)^T s_{CP} \, \|_2^2$.

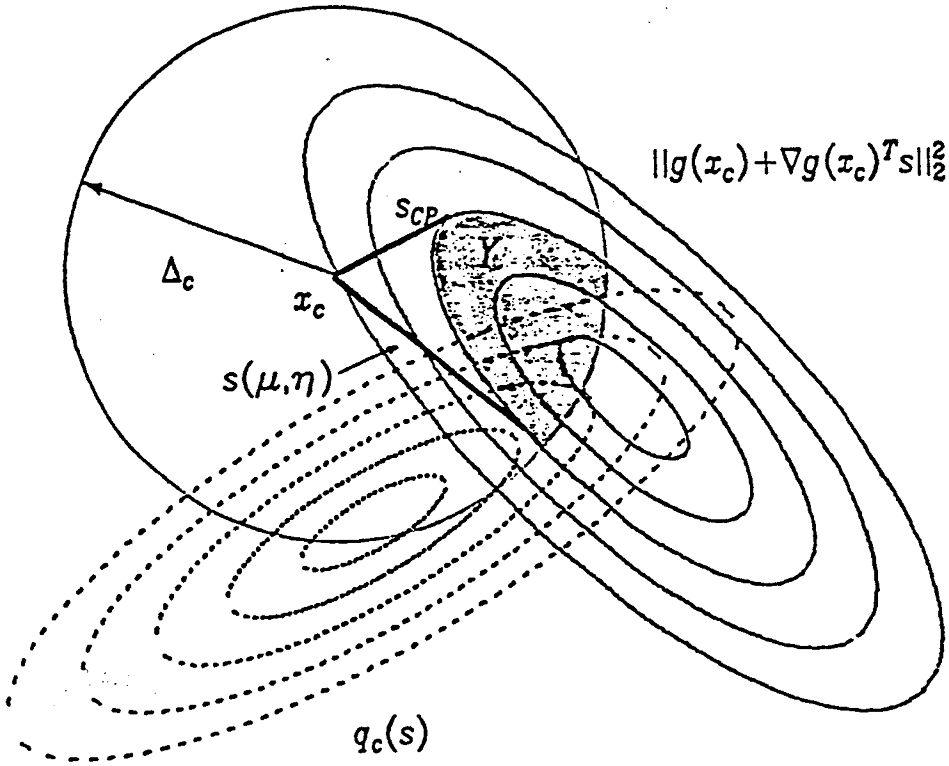

FIGURE 1

3. Theory. We consider the problem :

$$\text{(QPQ)} \qquad \begin{aligned} &\textit{minimize} \;\; q(s) = a^T s + \tfrac{1}{2} s^T B s \\ &\textit{subject to} \;\; \| s \|_2 \leqq \Delta \\ &\qquad\qquad\quad \| g(x) + \nabla g(x)^T s \|_2^2 \leqq \theta \,, \end{aligned}$$

where $a \in \mathrm{R}^n$ and $B \in \mathrm{R}^{n \times n}$ is symmetric and nonsingular. Problem QPQ is the basis of our trust region approach to equality constrained minimization. Its solution is given by the following lemma.

LEMMA 3.1 *Problem QPQ is solved by :*

$$s(\mu,\eta) \equiv -\left(B + \mu I + \eta \nabla g(x) \nabla g(x)^T\right)^{-1} \left(a + \nabla g(x) \eta g(x)\right)$$

for $\mu,\eta \geqq 0$ *such that* $\|s(\mu,\eta)\|_2 = \Delta$ *and* $\| g(x) + \nabla g(x)^T s \|_2^2 = \theta$, *unless:*

$\|g(x)+\nabla g(x)^T s(0,0)\|_2^2 < \theta$, *and* $\|s(0,0)\|_2^2 < \Delta$ *in which case* $s(0,0)$ *is the solution,*

or

$\|g(x)+\nabla g(x)^T s(\mu,0)\|_2^2 < \theta$, *and* $\|s(\mu,0)\|_2^2 = \Delta$ *in which case* $s(\mu,0)$ *is the solution,*

or

$\|g(x)+\nabla g(x)^T s(0,\eta)\|_2^2 = \theta$, *and* $\|s(0,\eta)\|_2^2 < \Delta$ *in which case* $s(0,\eta)$ *is the solution.*

Proof. The proof is a straightforward application of the necessary conditions of constrained optimization. □

By defining a and B in various ways we can show now the solution to problem QPQ is related to existing theory. The following theorem shows that if the quadratic model $q(s)$ is the Taylor expansion of f, and the trust region constraint is not binding, then our step is the Newton step on the standard penalty function with penalty constant η. It is important to note that η is not a free parameter, but is determined by the solution to problem QPQ.

THEOREM 3.1 *Let* $a = \nabla f(x)$ *and* $B = \nabla^2 f(x)$. *If* $\nabla^2 f(x)$ *is nonsingular and* Δ *is such that the constraint* $\|s\| \leqq \Delta$ *is not binding, then* $s(0,\eta)$ *is the Newton step for the standard penalty function*

$$P(x) = f(x) + \frac{1}{2}\eta g(x)^T g(x) .$$

Moreover, if $\nabla^2 f(x)$ *is positive definite, then for any* $\mu \geqq 0$, $s(\mu,\eta)$ *is a descent direction for* $P(x)$.

Proof. The proof of the first part is straightforward from the definition of the Newton step for minimizing a function. Details can be found in section 5.5 of Dennis and Schnabel [1]. In this case

$$s(0,\eta) = -\left(\nabla^2 f(x) + \eta \nabla g(x) \nabla g(x)^T\right)^{-1} \left(\nabla f(x) + \nabla g(x) \eta g(x)\right).$$

To prove that $s(\mu,\eta)$ is a descent direction for P, it is sufficient to show that $\nabla P(x)^T s(\mu,\eta) < 0$. Noting that μ and η are nonnegative, $\nabla^2 f(x)$ is positive definite, and $\nabla P(x) = \nabla f(x) + \nabla g(x) \eta g(x)$, we have

$$\nabla P(x)^T s(\mu,\eta) = -\nabla P(x)^T \left(\nabla^2 f(x) + \mu I + \eta \nabla g(x) \nabla g(x)^T\right)^{-1} \nabla P(x) < 0. \square$$

Now we show that if $q(s)$ is the Taylor expansion of the Lagrangian function, then the step that solves problem QPQ is the Newton step on the augmented Lagrangian function

$$AL(x,\lambda) = f(x) + \lambda^T g(x) + \frac{1}{2} \eta g(x)^T g(x).$$

Again, it is important to note that the penalty constant is determined by the solution to problem QPQ.

THEOREM 3.2 *Let $a = \nabla_x L(x,\lambda)$ and $B = \nabla^2_{xx} L(x,\lambda)$. If $\nabla^2_{xx} L(x,\lambda)$ is nonsingular and Δ is such that the constraint $\|s\| \leq \Delta$ is not binding, then $s(0,\eta)$ is the Newton step for the augmented Lagrangian. Moreover, if $\nabla^2_{xx} L(x,\lambda)$ is positive definite, then for ~~any~~ $\mu \geq 0$, $s(\mu,\eta)$ is a descent direction for $AL(x,\lambda)$.*

Proof. The proof is analogous to the proof of the previous theorem. $\square$

We have shown how our approach relates to the standard penalty function and the augmented Lagrangian. It is also possible to relate the solution to problem QPQ to s_{QP}, the solution of problem QP. We know

$$s_{QP} = -B^{-1} \left(\nabla f(x) + \nabla g(x) \lambda\right),$$

where

$$\lambda = \left(\nabla g(x)^T B^{-1} \nabla g(x)\right)^{-1} \left(g(x) - \nabla g(x)^T B^{-1} \nabla f(x)\right).$$

See Tapia [13] for details and background material. The following theorem shows that one should not expect the solutions of problems QPQ and QP to be the same. It is reasonable to compare solutions of the two problems only in the case that the trust region constraint in problem QPQ is not binding.

THEOREM 3.3 *Let* $a = \nabla_x L(x,\lambda)$, *and* Δ *be such that the constraint* $||s|| \leq \Delta$ *is not binding. Then the solution of problem QPQ is the solution to problem QP if and only if the unconstrained minimizer of the quadratic,* $q(s)$, *satisfies linearized constraints.*

Proof. First we will assume that $s_{QP} = s(\mu,\eta)$ and show that the unconstrained minimizer of $q(s)$ satisfies linearized constraints.

Since s_{QP} solves problem QP, it satisfies linearized constraints, i.e.,

$$g(x) + \nabla g(x)^T s_{QP} = 0 .$$

Given that $s(\mu,\eta) = s_{QP}$, we have

$$g(x) + \nabla g(x)^T s(\mu,\eta) = 0 .$$

If $\theta = 0$, we observe that problems QP and QPQ are equivalent, therefore, we consider $\theta > 0$. Since $\theta > 0$ the constraint

$$||g(x) + \nabla g(x)^T s||_2^2 \leq \theta ,$$

is not binding and the multiplier η associated with this constraint is 0. Since $||s|| \leq \Delta$, is assumed not to be a binding constraint, the solution to problems QP and QPQ is

$$s(0,0) = -B^{-1} (\nabla_x L(x,\bar{\lambda}))$$

which is the unconstrained minimizer of $q(s)$.

Next we must show that if the unconstrained minimizer of $q(s)$ satisfies linearized constraints, then $s_{QP} = s(\mu,\eta)$. The result follows from the fact that in this case both problems become the same unconstrained minimization problem. □

As we progress through the iterations of our algorithm we should expect to have Δ large and $\theta \to 0$. Clearly for Δ sufficiently large we will have $\mu = 0$. Also from Theorem 3.3 we are led to conjecture that $\eta \to \infty$ as $\theta \to 0$. Hence, we are interested in the behavior of the solution of problem QPQ as $\eta \to \infty$ and $\mu \to 0$. The following theorem gives us this behavior, which can be viewed as a form of consistency. Namely, while the solution of problem QP and problem QPQ are in general never the same; as $\eta \to \infty$ and $\mu \to 0$ the solution of problem QPQ approaches the solution of problem QP. Thus we should expect our algorithm to eventually generate steps which are arbitrarily close to the SQP step. In practice we have found this to be the case. These comments are the subject of the following theorem.

THEOREM 3.4 *Let* $a = \nabla_x L(x,\lambda)$, *and* B *be positive definite. Then*

$$\lim_{(\mu,\eta)\to(0,\infty)} s(\mu,\eta) = s_{QP}.$$

Proof. To prove this theorem we need to obtain $(B+\mu I+\eta\nabla g(x)\nabla g(x)^T)^{-1}$. By the Sherman-Morrison-Woodbury formula, see page 50 of Ortega and Rheinboldt [8], we have

$$(B+\mu I+\eta\nabla g(x)\nabla g(x)^T)^{-1} = (B+\mu I)^{-1} - \eta(B+\mu I)^{-1}\nabla g(x) [I+\eta\nabla g(x)(B+\mu I)^{-1}\nabla g(x)]^{-1}\nabla g(x)^T(B+\mu I)^{-1}.$$

Therefore,

$$\begin{aligned} s(\mu,\eta) &= -((B+\mu I)+\eta\nabla g(x)\nabla g(x)^T)^{-1}(\nabla_x L(x,\lambda)+\nabla g(x)\eta g(x)) \\ &= -(B+\mu I)^{-1}(I - \nabla g(x)[\tfrac{1}{\eta}I+\nabla g(x)^T(B+\mu I)^{-1}\nabla g(x)]^{-1}\nabla g(x)^T(B+\mu I)^{-1}) \\ &\quad (\nabla_x L(x,\lambda) + \nabla g(x)\eta g(x)) \\ &= -(B+\mu I)^{-1}(\nabla_x L(x,\lambda)+\nabla g(x)[\tfrac{1}{\eta}I+\nabla g(x)^T(B+\mu I)^{-1}\nabla g(x)]^{-1} \\ &\quad (g(x)-\nabla g(x)^T(B+\mu I)^{-1}\nabla_x L(x,\lambda))) \end{aligned}$$

Taking limits as $\eta\to\infty$ and $\mu\to 0$ we have

$$\begin{aligned} \lim_{(\mu,\eta)\to(0,\infty)} s(\mu,\eta) = & -B^{-1}(\nabla_x L(x,\lambda)+\nabla g(x)[\nabla g(x)^T B^{-1}\nabla g(x)]^{-1} \\ & (g(x)-\nabla g(x)^T B^{-1}\nabla_x L(x,\lambda))) . \end{aligned}$$

It is straightforward to see that by substituting $\nabla f(x)+\nabla g(x)\lambda$ for $\nabla_x L(x,\lambda)$ we obtain s_{QP}. □

4. Numerical Results. In order to study the effectiveness of our approach from arbitrary starting points, we produced a preliminary implementation. Problem QPQ was solved by a modification of the iterative process that was first suggested for nonlinear least squares by

Hebden [4] and Moré [6]. For our quadratic objective function we choose $q(s) = \nabla f(x)^T s + \frac{1}{2} s^T \nabla^2 f(x) s$ with no multiplier approximations. Although the algorithm is not completely defined, we wanted to obtain some feel for the robustness of the approach. For this we compared our method, SQPQC, with an SQP approach, VF02AD by Powell [9], which is available in the Harwell Subroutine Library.

We now list a subset of our test problems. These problems are referenced and can be found in Hock and Schittkowski [5]. The number in parentheses denotes the number given to this problem in [5], n is the number of variables in the problem, and m is the number of equality constraints.

Problem 1 (60) $n = 3,\ m = 1$

$$f(x) = (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^4$$

$$g_1(x) = x_1(1 + x_2^2) + x_3^4 - 4 - 3(2)^{\frac{1}{2}}$$

$$x_* \approx (1.1048,\ 1.1966,\ 1.5352)$$

Problem 2 (77) $n = 5,\ m = 2$

$$f(x) = (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_4 - 1)^4 + (x_5 - 1)^6$$

$$g_1(x) = x_1^2 x_4 + \sin(x_4 - x_5) - 2(2)^{\frac{1}{2}}$$

$$g_2(x) = x_2 + x_3^4 x_4^2 - 8 - (2)^{\frac{1}{2}}$$

$$x_* \approx (1.1661,\ 1.1821,\ 1.3802,\ 1.5060,\ 0.6109)$$

Problem 3 (79) $n = 5,\ m = 3$

$$f(x) = (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_4 - 1)^4 + (x_5 - 1)^4$$

$$g_1(x) = x_1 + x_2^2 + x_3^3 - 2 - 3(2)^{\frac{1}{2}}$$

$$g_2(x) = x_2 + x_3^2 + x_4 + 2 - 2(2)^{\frac{1}{2}}$$

$$g_3(x) = x_1 x_5 - 2$$

$$x_* \approx (1.1911,\ 1.3626,\ 1.4728,\ 1.6350,\ 1.6790)$$

Problem 4 (78) $\quad n = 5, \ m = 3$

$$f(x) = x_1 x_2 x_3 x_4 x_5$$

$$g_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10$$

$$g_2(x) = x_2 x_3 - 5 x_4 x_5$$

$$g_3(x) = x_1^3 + x_2^3 + 1$$

$$x_* \approx (-1.7171, 1.5957, 1.8272, -0.7636, -0.7636)$$

The results from this subset of test problems are reported in Table 1. The column labeled Convergence indicates whether or not convergence was obtained, and the number in parentheses indicates the number of iterations the algorithm took to converge. This number does not give meaningful comparisons for many reasons, including the fact that the algorithm is only in a preliminary stage. We have, however, included it for completeness.

Although the number of problems is small, it can be seen that SQPQC converges for all the problems that VF02AD converges. We have found several problems where the linesearch routine in VF02AD fails, and thus halts, but our trust region routine is successful. For example, problem 2 with starting point (10, 10, 10, 10, 10). At the first iteration in VF02AD, the line search routine fails to locate a better point. Whereas, our trust region routine succeeds in finding a next iterate and proceeds to find the solution.

5. Concluding Remarks. We have presented a framework for a trust region approach for solving equality constrained optimization problems. At each iteration the subproblem we solve is not in general the successive quadratic programming, (SQP), subproblem. We have motivated the conjecture that asymptotically our step is the same as the step produced by solving the SQP subproblem.

The theoretical results presented in this paper, although preliminary, have established important links between the step selection process and several widely used merit functions. We have shown that the step we obtain is a descent direction on either the standard penalty function or the augmented Lagrangian function, where each penalty constant is provided by the solution to the associated subproblem.

A preliminary implementation of our approach has produced good numerical results. These numerical results, and the preliminary theory,

Problem	Starting Point	Convergence (No. of Iterations) VF02AD	SQPQC
1	(1.5,1.5,1.5)	Y (6)	Y (16)
1	(1,2,3)	Y (10)	Y (17)
1	(1.4,1.5,1.9)	Y (7)	Y (16)
1	(11,12,15)	Y (19)	Y (24)
1	(2.7,2.9,3.8)	Y (10)	Y (19)
1	(27,29,38)	Y (36)	Y (31)
1	(10,10,10)	Y (17)	Y (23)
2	(1,1,1,1,1)	Y (13)	Y (17)
2	(10,10,10,10,10)	N (*)	Y (22)
2	(2,2,2,2,2)	Y (15)	Y (11)
2	(-1,3,-0.5,-2,-3)	N (*)	N (*)
2	(-3,-3,3,9,0)	Y (21)	Y (38)
2	(-1,8,3,3,0)	N (*)	Y (31)
2	(4,3,7,-5,-3)	N (*)	N (*)
3	(-1,3,-0.5,-2,-3)	Y (10)	Y (10)
3	(-1,2,1,-2,-2)	Y (16)	Y (30)
3	(1,1,1,1,1)	Y (8)	Y (6)
3	(10,10,10,10,10)	Y (18)	Y (31)
3	(2,2,2,2,2)	Y (9)	Y (7)
3	(-2,-2,-2,-2,-2)	Y (16)	Y (25)
4	(-1,1.5,2,-1,-2)	Y (10)	Y (11)
4	(-10,10,10,-10,-10)	Y (21)	Y (9)
4	(-1,2,1,-2,-2)	Y (9)	Y (5)
4	(-1,-1,-1,-1,-1)	Y (9)	Y (5)
4	(-2,2,2,2,2)	Y (7)	Y (5)

TABLE 1

lead us to believe that our approach is worthy of continued research.

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